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From Perplexity to Mathematical Certainty: Reflective Clarificative Thinking in Solving Systems of Linear Equations

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ABSTRACT

Solving systems of linear equations may generate doubt that requires scrutiny of information, procedures, and results. This study aims to describe and reconstruct pre-service mathematics teachers' reflective clarificative thinking in addressing perplexity across Polya's four problem-solving stages. A qualitative multiple-case study involved two purposively selected participants who solved the same traffic-flow conservation problem using substitution. Data were collected through a problem-solving test and in-depth task-based interviews, including a follow-up interview on S2's solution checking, and examined through within-case and cross-case analyses. The findings show that S1 engaged in confirmatory clarification by verifying the equations to establish solution correctness. S2 engaged in corrective clarification by detecting an inconsistency, tracing a subtraction error, correcting the initial value and its dependent variables, and verifying the revised solution without changing the substitution method. Both forms of clarification supported mathematically justified certainty. This study contributes a distinction between confirmatory and corrective clarification as responses to doubt during problem solving. These findings highlight the importance of learning activities that encourage pre-service teachers to articulate uncertainty, examine sources of error, explain corrections, and verify solutions against all problem conditions, thereby strengthening their capacity to justify solutions and make mathematical judgments grounded in careful reasoning that can be substantiated.

Keywords: Reflective Clarificative Thinking, Perplexity, Systems of Linear Equations, Problem Solving, Pre-Service Mathematics Teachers.

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INTRODUCTION

Problem solving is an essential component of mathematics learning, involving conceptual understanding, strategy selection, procedural implementation, and evaluation of solutions (Artigue & Houdement, 2007; Schoenfeld, 2020). For pre-service mathematics teachers, these abilities are also necessary for understanding and teaching problem solving meaningfully. (Barham, 2020) showed that problem-solving courses can broaden prospective teachers' repertoires of strategies, while (Giyarti, Putranto, & Sofyan Tio Fajar Maulana, 2025) highlighted the importance of examining their understanding of mathematical problems and their teaching. However, problem solving does not necessarily proceed directly or linearly; it may involve revisiting interpretations, plans, and solution steps (Rott, Specht, & Knipping, 2021a). Conceptual and procedural knowledge therefore needs to be accompanied by the ability to question and verify the reasoning underlying a solution.

Systems of linear equations provide a relevant context for examining these activities because they involve interpreting symbols, relating variables, and coordinating representations with algebraic procedures. (Carlson, 1993) emphasized the importance of conceptual understanding in linear algebra education. In contextual problems, successful solutions also depend on accurately interpreting information before performing mathematical operations (Boonen, de Koning, Jolles, & van der Schoot, 2016). A traffic-flow problem, for example, requires students to interpret flow directions, formulate equations using flow conservation, select a substitution sequence, and verify the solution. Coordinating the diagram, equations, and calculations requires checking consistency across representations, as discussed in (Ainsworth, 2006) framework for learning with multiple representations. Misinterpreting an arrow or making an arithmetic error may produce an inconsistency that requires further examination.

Uncertainty during problem solving can be understood through the concept of perplexity: a state of confusion, doubt, or uncertainty about a situation. (Dewey, 1933) described reflective thinking as the active and careful consideration of beliefs or knowledge in light of their supporting grounds. (Rodgers, 2002) further characterized reflection as a systematic, inquiry-based process of meaning making. In mathematical problem solving, perplexity may prompt students to question information, procedures, and results. Nevertheless, doubt alone does not establish that reflection has occurred; the analysis must identify the actions taken to obtain clarity and the mathematical reasons underlying those actions.

In this study, reflective clarificative thinking is operationally defined as a process that begins with perplexity concerning information, concepts, procedures, or results and addresses it by reviewing the uncertain aspect and checking it against relevant mathematical grounds. This definition was developed for the present analysis, drawing on (Dewey, 1933; Rodgers, 2002) accounts of reflective inquiry and (Kaune, 2006) differentiation of reflective activities in mathematics. Clarification seeks clarity and confidence grounded in mathematical reasoning. Its defining features include the source of uncertainty, the checking activity, its mathematical basis, and its consequences for the solution, rather than merely a participant's statement of confidence.

This process is examined through four problem-solving stages: understanding the problem, devising a plan, carrying out the plan, and looking back (Polya, 1973). During understanding, clarification may concern the interpretation of information and the adequacy of the model. During planning, it may concern the sequence of steps and relationships among equations. During implementation, students may check operations and intermediate results, whereas during looking back they compare the solution with the original equations. These stages organize the analysis without implying that thinking necessarily follows a rigid sequence, given the transitions and revisiting of phases observed in problem solving (Rott et al., 2021a).

Previous studies provide a foundation for understanding the relationships among reflection, metacognition, and problem solving. (John H Flavell, 1979) discussed metacognition and cognitive monitoring, while (Schneider & Artelt, 2010) examined their role in mathematics education. (Jacobse & Harskamp, 2012) emphasized the importance of measuring metacognition through evidence of problem-solving activity, and (Geng, 2011) discussed self-monitoring in directing and correcting solutions. (Aras, Pratiwi, Nanna, & Barumbun, 2022) investigated scaffolding for prospective teachers' reflective thinking. Recent reviews have also examined metacognition and problem-solving ability (Nur, Ajizah, & Mariani, 2026), technological support for metacognition (Mannan, Bajuri, Abdul Rahim, & Abdullah, 2026), and the development of prospective teachers' mathematical thinking (Mafarja, Rahim, & Zulnaidi, 2026).

The present study addresses the need for a detailed empirical account of how perplexity develops into clarification, which mathematical grounds are used, and how checking produces either certainty or a need for correction. This focus elaborates the relationship between reflection and metacognitive monitoring discussed by (Kaune, 2006) and (Jacobse & Harskamp, 2012), without claiming that reflective thinking in solving linear systems has not

previously been investigated. Written work is combined with task-based interviews to examine reasons and sources of doubt that may not be evident in a final answer. Comparing two participants who use substitution to solve the same problem allows the analysis to focus on variations in clarification.

This study aims to describe and reconstruct pre-service mathematics teachers' reflective clarificative thinking in addressing perplexity across the stages of solving a system of linear equations, using Polya's framework. Specifically, it seeks to identify sources of perplexity, explain the clarification activities used to address them, and reconstruct the movement from uncertainty toward mathematical clarity and confidence. The intended contribution is a more detailed understanding of clarification mechanisms in mathematical problem solving. The findings may also inform learning opportunities that encourage students to identify doubts, revisit mathematical reasons, and verify both their procedures and their solutions.

METHOD

A qualitative multiple-case study was conducted to describe and reconstruct pre-service mathematics teachers' reflective clarificative thinking in solving a system of linear equations. Two participants solved the same problem using substitution. This design enabled an in-depth examination of each participant's thinking and a cross-case comparison of similarities and variations in the mechanisms used to clarify perplexity. The analysis was organized around Polya's four stages: understanding the problem, devising a plan, carrying out the plan, and looking back.

The study involved students enrolled in a Mathematics Education program who had studied systems of linear equations in an Elementary Linear Algebra course. Two participants, S1 and S2, were purposively selected on the basis of a problem-solving test and preliminary interviews. Both exhibited features of reflective clarificative thinking: they experienced perplexity and sought to address it by clarifying relevant mathematical information, concepts, principles, or procedures. Both used substitution to solve the same problem, allowing the analysis to focus on reflection rather than differences in solution methods.

Data were collected through a problem-solving test and in-depth task-based interviews. The test presented a contextual problem involving traffic-flow conservation at four intersections. Participants were required to interpret information, formulate a system of linear equations, determine the unknown values, and verify the solution. Semi-structured interviews referred to each participant's written work to elicit sources of perplexity, clarification activities, mathematical reasons, and changes in understanding or confidence. The test task is presented in Figure 1. A follow-up interview with S2 explored the detection of inconsistency, the explanation of the error, recalculation, and solution verification. Actions evident in the initial written work were distinguished from corrective actions explained in the interview.

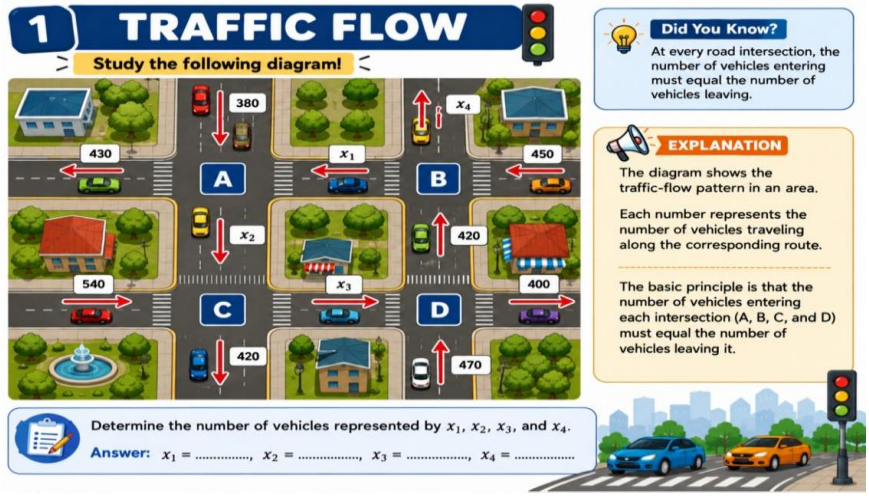


Figure 1. Traffic-flow problem-solving task.

Reflective clarificative thinking was analyzed using four indicators aligned with Polya's stages: KL1, clarification of problem information; KL2, clarification of the solution plan; KL3, clarification of solution procedures; and KL4, clarification of solution validity. For each indicator, the analysis examined the source of perplexity, the clarification activity, its mathematical basis, and its outcome. The indicators are presented in Table 1.

Table 1. Indicators of reflective clarificative thinking across Polya's problem-solving stages.

Polya stage	Code	Indikator
Understanding the problem	KL1	Identifies uncertainty about problem information and clarifies it by re-examining the information and relevant mathematical principles.
Devising a plan	KL2	Identifies uncertainty about solution steps and clarifies it by checking the plan against the mathematical structure already established.
Carrying out the plan	KL3	Identifies uncertainty about operations or calculated values and clarifies it by re-examining the procedures and mathematical relationships used.
Looking back	KL4	Clarifies solution validity by checking the results against the original equations and problem conditions to establish confidence grounded in mathematical evidence.

Data analysis comprised within-case and cross-case analyses. Within-case analysis examined S1's and S2's written work and interviews separately using Polya's four stages. Each stage was examined for perplexity, the source of doubt, clarification activities, and reflective outcomes. Cross-case analysis then compared the participants to identify similarities and variations in reflective clarificative thinking despite their use of substitution for the same problem. The resulting patterns were reconstructed from evidence in the two cases. The credibility of the interpretations was strengthened by comparing written work with interview accounts and checking the consistency between the data and the coding.

RESULT AND DISCUSSION

S1's Reflective Clarificative Thinking

Analysis of S1's written work and interview revealed uncertainty at several points, from interpreting the diagram to checking the final solution. This uncertainty prompted S1 to revisit the information, equation structure, and principles of algebraic operations to verify the solution process. S1's written work is presented below.

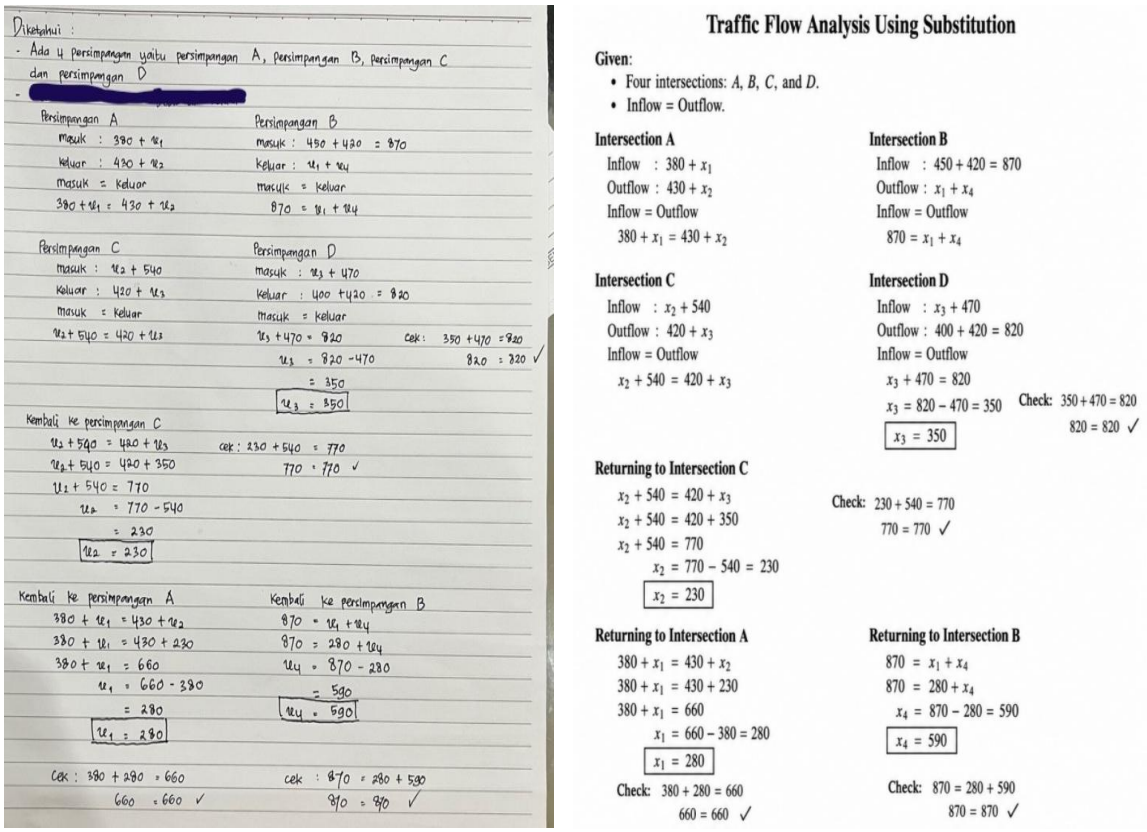


Figure 2. S1's written solution.

Understanding the Problem

S1 identified four intersections, A, B, C, and D, and four unknown variables, x_1, x_2, x_3, and x_4. S1 understood that the number of vehicles entering each intersection must equal the number leaving it. However, S1 experienced perplexity when interpreting flow directions, particularly at intersection D. This uncertainty was expressed in the following interview excerpt. R denotes the researcher.

R : Were you immediately certain about the flow directions you read?
S1: At first, I was still confused. I was somewhat unsure about the direction of the arrows at the intersection.
R : Which part made you unsure?
S1: Mainly at point D. I was unsure whether I should treat 400 and 420 as incoming flows, because they might actually be outgoing flows.
S1: I looked again at the arrows at point D in the diagram. I checked where the arrowheads were pointing.

The excerpt indicates that S1's uncertainty arose while interpreting the visual representation. To address it, S1 re-examined the arrows and checked them against the principle of traffic-flow conservation. This activity clarified the information before the equations were formulated.

S1's written work contained the following four equations:

$$380 + x_1 = 430 + x_2 \quad (1)$$
$$870 = x_1 + x_4 \quad (2)$$
$$x_2 + 540 = 420 + x_3 \quad (3)$$
$$x_3 + 470 = 820 \quad (4)$$

Reflective clarificative thinking at this stage involved doubt about flow directions, re-examination of the diagram, comparison with the conservation principle, and formulation of the corresponding equations.

Devising a Plan

S1 was unsure which equation to solve first. Although all four equations had been formulated, S1 did not immediately establish a solution sequence. This was evident in the following interview excerpt.

R : Did you immediately know which equation to use?
S1: No. I was initially confused about where to start.
R : What did you do?
S1: I looked at the equations again. I checked which equation would allow me to determine a variable more directly.
R : What did you find?
S1: The equation at intersection D is $470 + x_3 = 820$. Only x_3 is unknown there, so I can find x_3.

This uncertainty prompted S1 to re-examine the equation structure and the relationships among the variables. S1 identified that the equation for intersection D contained only one unknown and could therefore be solved first. After this clarification, S1 established the following sequence: $x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x_4$. Clarification thus helped S1 establish the order in which the equations should be used. The strategy remained unchanged; the substitution plan was made clearer through the relationships among the equations.

Carrying Out the Plan

S1 implemented substitution step by step, beginning with equation D to obtain $x_3 = 350$, followed by $x_2 = 230$, $x_1 = 280$, and $x_4 = 590$. During implementation, S1 expressed concern about possible errors in algebraic operations. Rather than immediately accepting $x_3 = 350$, S1 substituted this value into the original equation.

R : Are you certain that your answer, $x_3 = 350$, is correct?
S1: I checked my answer again. I substituted 350 into $x_3 + 470 = 820$ and obtained $350 + 470 = 820$. The right-hand side equals the left-hand side.
R : What made you want to check it again?
S1: I was worried that I might have made a mistake in the solution steps or in subtracting $820 - 470$.

Clarification also occurred when S1 determined x_2. S1 briefly questioned which operation to use, then revisited the structure of equation C before continuing the calculation. S1 therefore used the original equations as a reference for checking operations and intermediate results. After checking, S1 continued until all variable values had been obtained.

Looking Back

S1 substituted all variable values into the four equations to verify that each satisfied the principle of flow conservation.

Intersection	Verification
A	380 + 280 = 430 + 230 = 660
B	870 = 280 + 590
C	230 + 540 = 420 + 350 = 770
D	350 + 470 = 820

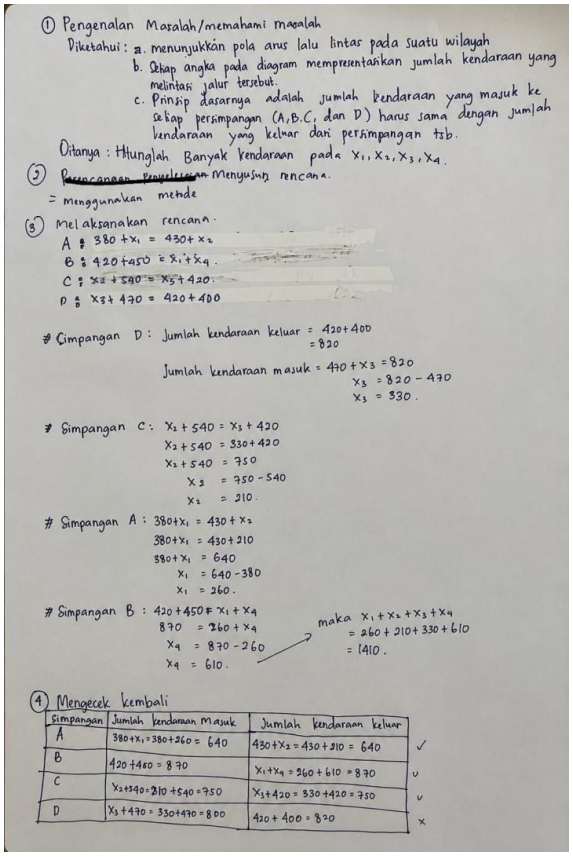
The checks showed that all four equations were satisfied. S1 also explained why every equation needed to be checked.

R : Why did you need to check all the equations again?
S1: I wanted to make sure there were no errors in the earlier steps. If I checked only one equation, an error in another equation might go unnoticed.
R : After checking all of them, how did you feel about your answer?
S1: I was more confident that the answer was correct.

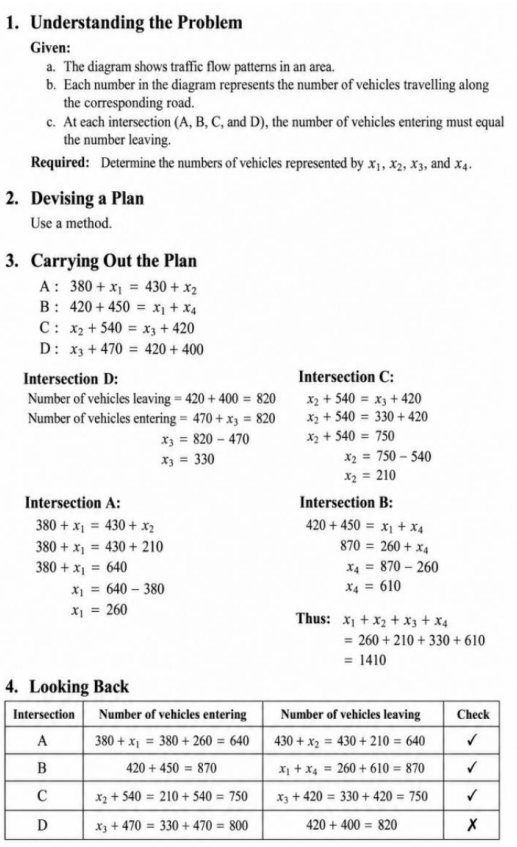
These findings indicate that S1 checked the solution both to verify the final values and to establish that all mathematical relationships were satisfied. Clarification strengthened confidence without changing either the method or the solution. Overall, S1's reflective clarificative thinking followed a pattern of uncertainty, reconsideration of information or procedures, comparison with mathematical concepts, and confirmation of correctness. Its primary function was to establish the validity of the process and the solution.

S2's Reflective Clarificative Thinking

Analysis of S2's written work and follow-up interview showed that sequential substitution initially produced an answer containing a subtraction error. During looking back, S2 detected an inconsistency, traced its cause, recalculated the variables dependent on the incorrect value, and verified the revised solution. S2's initial written work is presented below; the subsequent account of correction draws on the follow-up interview.


① Pengenalan Masalah/memahami masalah
Diketahui : a. menunjukkan pola arus lalu lintas pada suatu wilayah
b. Setiap angka pada diagram mempresentasikan jumlah kendaraan yang melintasi jalur tersebut.
c. Prinsip dasarnya adalah jumlah kendaraan yang masuk ke setiap persimpangan (A,B,C, dan D) harus sama dengan jumlah kendaraan yang keluar dari persimpangan tsb.
Ditanya : Hitunglah Banyak kendaraan pada x_1, x_2, x_3, x_4.
② Menyusun rencana.
= menggunakan metode
③ Melaksanakan rencana.
A : $380 + x_1 = 430 + x_2$
B : $420 + 450 = x_1 + x_4$.
C : $x_2 + 540 = x_3 + 420$.
D : $x_3 + 470 = 420 + 400$
Simpangan D : Jumlah kendaraan keluar = $420 + 400 = 820$
Jumlah kendaraan masuk = $470 + x_3 = 820$
$x_3 = 820 - 470$
$x_3 = 330$.
Simpangan C : $x_2 + 540 = x_3 + 420$
$x_2 + 540 = 330 + 420$
$x_2 + 540 = 750$
$x_2 = 750 - 540$
$x_2 = 210$.
Simpangan A : $380 + x_1 = 430 + x_2$
$380 + x_1 = 430 + 210$
$380 + x_1 = 640$
$x_1 = 640 - 380$
$x_1 = 260$.
Simpangan B : $420 + 450 = x_1 + x_4$
$870 = 260 + x_4$
$x_4 = 870 - 260$
$x_4 = 610$.
maka $x_1 + x_2 + x_3 + x_4 = 260 + 210 + 330 + 610 = 1410$.
④ Mengecek kembali

Simpangan	Jumlah kendaraan masuk	Jumlah kendaraan keluar	
A	$380 + x_1 = 380 + 260 = 640$	$430 + x_2 = 430 + 210 = 640$	✓
B	$420 + 450 = 870$	$x_1 + x_4 = 260 + 610 = 870$	✓
C	$x_2 + 540 = 210 + 540 = 750$	$x_3 + 420 = 330 + 420 = 750$	✓
D	$x_3 + 470 = 330 + 470 = 800$	$420 + 400 = 820$	✗


1. Understanding the Problem

Given:

a. The diagram shows traffic flow patterns in an area.
b. Each number in the diagram represents the number of vehicles travelling along the corresponding road.
c. At each intersection (A, B, C, and D), the number of vehicles entering must equal the number leaving.

Required: Determine the numbers of vehicles represented by x_1, x_2, x_3, and x_4.

2. Devising a Plan

Use a method.

3. Carrying Out the Plan

A : $380 + x_1 = 430 + x_2$
B : $420 + 450 = x_1 + x_4$
C : $x_2 + 540 = x_3 + 420$
D : $x_3 + 470 = 420 + 400$

Intersection D:
Number of vehicles leaving = $420 + 400 = 820$
Number of vehicles entering = $470 + x_3 = 820$
$x_3 = 820 - 470$
$x_3 = 330$

Intersection C:
$x_2 + 540 = x_3 + 420$
$x_2 + 540 = 330 + 420$
$x_2 + 540 = 750$
$x_2 = 750 - 540$
$x_2 = 210$

Intersection A:
$380 + x_1 = 430 + x_2$
$380 + x_1 = 430 + 210$
$380 + x_1 = 640$
$x_1 = 640 - 380$
$x_1 = 260$

Intersection B:
$420 + 450 = x_1 + x_4$
$870 = 260 + x_4$
$x_4 = 870 - 260$
$x_4 = 610$

Thus: $x_1 + x_2 + x_3 + x_4 = 260 + 210 + 330 + 610 = 1410$

4. Looking Back

Intersection	Number of vehicles entering	Number of vehicles leaving	Check
A	$380 + x_1 = 380 + 260 = 640$	$430 + x_2 = 430 + 210 = 640$	✓
B	$420 + 450 = 870$	$x_1 + x_4 = 260 + 610 = 870$	✓
C	$x_2 + 540 = 210 + 540 = 750$	$x_3 + 420 = 330 + 420 = 750$	✓
D	$x_3 + 470 = 330 + 470 = 800$	$420 + 400 = 820$	✗

Figure 3. S2's initial written solution.

Understanding the Problem

S2 identified the four intersections, the flow directions, and the four unknown variables. S2 understood that incoming and outgoing vehicle flows must be equal at each intersection. Nevertheless, S2 was uncertain about interpreting flow directions in the diagram.

R : When you first read the problem, was there any information that you found unclear or were unsure about?
S2: Yes. I first had to make sure which vehicles were entering the intersection and which were leaving it in the diagram.
R : How did you address that uncertainty?
S2: I looked at the arrows again and checked them against the statement that the incoming flow must equal the outgoing flow.

This account shows that S2 recognized the possibility that misinterpreting the visual information could affect the mathematical model. S2 clarified the flow directions by comparing the diagram with the conservation principle. Through this activity, S2 clarified the given information, identified the unknown variables, and formulated the equations used to solve the problem.

Devising a Plan

After understanding the problem, S2 planned to use substitution. Equation D was selected as the starting point because it allowed one variable to be determined directly. The resulting value was then used to solve equations C, A, and B in sequence. Clarification at this stage was particularly evident when S2 checked the consistency between the equations and the diagram.

R : Were you uncertain at any point when formulating the equations?
S2: Yes. I had to make sure which flows were entering and which were leaving so that I would not place the variables incorrectly.
R : How did you know that your equations represented the diagram?
S2: I matched each term in the equations to the corresponding vehicle flow in the diagram.

S2 thus clarified the adequacy of the model before performing the calculations. The equation structure established the solution sequence.

Carrying Out the Plan

S2 used sequential substitution. The initial written solution gave: $x_1 = 260$, $x_2 = 210$, $x_3 = 330$, and $x_4 = 610$. S2 obtained these values step by step, starting with equation D and continuing with equations C, A, and B. During the calculations, S2 attempted to check the algebraic operations by revisiting the equation forms and the concept of inverse operations. However, the written work revealed a calculation error in determining x_3. Because substitution was performed sequentially, this error affected the subsequent variable values. Thus, although S2 selected an appropriate strategy, the arithmetic implementation did not initially produce a solution satisfying all the conditions of the problem.

Looking Back

S2 substituted the initial values, $x_1 = 260$, $x_2 = 210$, $x_3 = 330$, and $x_4 = 610$, into the equations for each intersection. Equations A, B, and C were satisfied, but equation D gave 330 + 470 = 800 rather than the required value of 820. This inconsistency generated perplexity concerning S2's initial confidence. The following excerpts from the follow-up interview illustrate the clarification process.

a. Recognizing an Inconsistency

R : What did you find when you checked your answer?
S2: The results for intersections A, B, and C were consistent, Ma'am, but the two sides were not equal at intersection D.
R : What did you think when you found that?
S2: I began to doubt my answer, Ma'am. At first, I thought that all the values were correct, but it turned out that one equation was not satisfied.
R : Why did that make you doubt your answer?
S2: Because the incoming and outgoing flows should be equal at every intersection. If they are different, there must be an error in my solution.

S2's doubt concerned a specific issue: the solution did not satisfy equation D. The conservation principle provided a reason to reconsider the solution. Initial confidence was therefore challenged by mathematical evidence inconsistent with the answer.

b. Tracing the Source of the Error

R : What did you do after finding that equation D was not satisfied?
S2: I looked at equation D again and checked the calculation I had used to find x_3.
R : Why did you check the calculation of x_3 first?
S2: Because I obtained x_3 from equation D, and I used that value to find the other variables.
R : Please calculate the subtraction again.
S2: 820 − 470 = 350, Ma'am.
R : What about your earlier result, $x_3 = 330$?
S2: It turns out that I made a subtraction error, Ma'am. It should have been 350, but I had written 330.
R : How did you confirm that 350 was correct?
S2: I substituted 350 into equation D and obtained 350 + 470 = 820. The result equals the right-hand side, so $x_3 = 350$ is correct.

The interview indicates that S2 located the error in the subtraction rather than in the conservation principle or the choice of method. S2 revisited the original equation, recalculated 820 − 470, and checked the result by addition. This examination clarified both the cause of the inconsistency and the value requiring correction.

c. Correcting the Sequential Substitution Results

R : Once you knew that x_3 should be 350, was correcting x_3 alone sufficient?
S2: No, Ma'am. I used x_3 to find x_2, then x_2 to find x_1, and x_1 to find x_4.
R : What did you need to do, then?
S2: I had to recalculate the other variables using the correct value of x_3.
R : What values did you obtain after making the corrections?
S2: I obtained $x_1 = 280$, $x_2 = 230$, $x_3 = 350$, and $x_4 = 590$.
R : Did you use a different solution method to correct your answer?
S2: No, Ma'am. I still used substitution. I only corrected the calculation error and recalculated the variables that depended on that result.

In the interview, S2 explained the recalculation sequence: $x_3 = 820 - 470 = 350$; $x_2 = 420 + 350 - 540 = 230$; $x_1 = 430 + 230 - 380 = 280$; and $x_4 = 870 - 280 = 590$. The correction encompassed every value affected by the initial error. S2 retained the substitution strategy while correcting its computational implementation.

d. Verifying the Entire Revised Solution

R : After obtaining the new values, were you certain that your answer was correct?
S2: I needed to check it again, Ma'am, because I had also thought that my earlier answer was correct, but it contained an error.
R : How did you check it?
S2: I substituted all the corrected values into the equation for each intersection.
R : What could you conclude from those checks?
S2: All the corrected variable values satisfied the equations at every intersection. Now I was confident that the result was correct.
R : What did you learn from the error in your earlier answer?
S2: I realized that an error in one calculation could affect the results that followed. Therefore, I needed to check the results against the original equations, rather than simply see whether I had completed all the solution steps.

S2 explained the checks at all four intersections: A, 380 + 280 = 430 + 230 = 660; B, 870 = 280 + 590; C, 230 + 540 = 420 + 350 = 770; and D, 350 + 470 = 820. Every equation was satisfied. Final confidence was grounded in verification of the entire revised solution, rather than merely in completion of the recalculation.

This sequence illustrates corrective clarification: inconsistency generated doubt; doubt was addressed by tracing its cause; the incorrect value and the dependent results were corrected; and every equation was verified. S2 explained this tracing and correction in the follow-up interview. The correction was therefore supported by the participant's account rather than solely by the researcher's mathematical reconstruction. Whereas checking confirmed S1's initially correct answer, it led S2 to revise the answer before attaining certainty.

Across the two cases, reflective clarificative thinking was evident within Polya's four stages. Although both participants used substitution, they differed in the aspects that prompted uncertainty, their checking activities, and the outcomes of clarification.

Table 2. Reflective clarificative thinking across Polya's four problem-solving stages.

Polya stage	S1	S2
Understanding the problem	Clarifies flow directions	Clarifies flow directions and symbol meanings
Devising a plan	Establishes the equation sequence	Checks model consistency
Carrying out the plan	Checks operations and intermediate results	Checks operations, but makes a calculation error
Looking back	Verifies that all equations are satisfied	Detects inconsistency, traces the error, recalculates, and verifies all equations

Consequence of clarification	Confirmation of correctness	Correction and certainty following renewed verification

Clarification had different consequences in the two cases. S1 attained certainty after verifying a solution that was correct from the outset. S2 detected an inconsistency, explained the subtraction error, recalculated all variables through sequential substitution, and attained certainty after checking the revised solution. For S2, looking back involved revisiting the calculations, giving the solution process an iterative character.

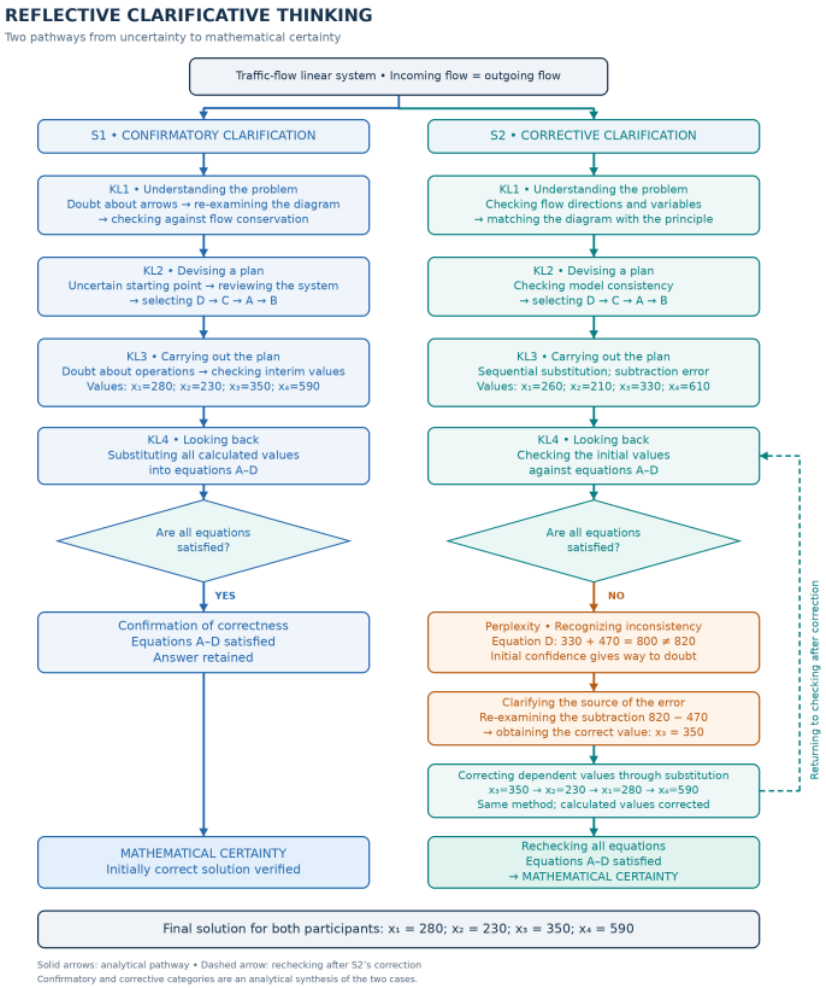


Figure 4. Contrasting pathways of reflective clarificative thinking in S1 and S2.

The cases therefore illustrate two consequences of clarification: confirmatory clarification and corrective clarification. These terms are analytical categories developed through the case comparison.

DISCUSSION

The two cases indicate that reflective clarificative thinking involved examining a specific doubt against mathematical grounds and using the outcome to confirm or revise a solution. S1 confirmed an initially correct answer, whereas S2 found an arithmetic error, recalculated dependent values, and verified the revised answer. This interpretation is consistent with (Dewey, 1933) account of reflective inquiry and (Rodgers, 2002) emphasis on systematic meaning making. It also gives a task-specific account of the reflective activities discussed by (Kaune, 2006) : confidence became warranted through an explanation of why the equations and calculated values were consistent.

The emergence of perplexity aligns with (Hidajat, Sa'dijah, Sudirma, & Susisw, 2019), who used two school-student cases to identify sources of perplexity through arguments about unfamiliar geometry problems. Both studies show the value of examining learners' explanations of uncertainty. However, the present cases concerned interpretation and checking within a familiar algebraic method, including a subtraction error, rather than difficulty generating a new strategy. Similarly, (Aras et al., 2022) found that questioning and evaluation could require scaffolding before prospective teachers progressed to reasoning. Here, S2's follow-up interview elicited an account of correction, but substitution was retained. Differences in task novelty and the purposes of researcher questioning may help explain these variations; the present evidence does not establish that S2 corrected the solution independently of prompts.

During understanding, clarification connected the arrow directions with the conservation principle and the terms in each equation. This accords with (Zulu & Mudaly, 2023) finding that visualization and self-directed discourse supported prospective mathematics teachers' problem solving. It is also relevant to (Nanna & Pratiwi, 2020) identification of cognitive barriers in picture-based fraction tasks. Both studies highlight the interpretive demands of visual information, although the present participants resolved their uncertainty by matching flows to equations. (Ainsworth, 2006) framework helps explain this consistency: learners must interpret individual representations and their relationships. (Boonen, Reed, Schoonenboom, & Jolles, 2016) likewise emphasize the importance of understanding contextual information before calculation. Different visual conventions, mathematical topics, and prior knowledge may influence which interpretive difficulties emerge.

Planning findings extend earlier work on prospective teachers' strategies. (Barham, 2020) documented a broader strategy repertoire following problem-solving lessons, while still identifying weaknesses in comprehension and mathematical work. (Giyarti et al., 2025; Jiang, Zhang, Jiang, & Xiong, 2022) examined prospective teachers' understanding or perceptions of problem solving and its teaching. The present study shares their concern with reasoning beyond routine execution, but examines how two participants scrutinized a plan while using the same method. S1 clarified the starting equation and substitution sequence; S2 checked model consistency. Retaining substitution may reflect the task's accessible one-unknown equation and the participants' prior study of linear systems. These are plausible contextual explanations, not measured effects of prior instruction.

The movement between checking and recalculation is consistent with (Melanie G. Gurat, 2016) cyclic account of metacognitive strategy knowledge among pre-service teachers. Their framework links preparation, production, and evaluation, whereas the present analysis tracks specific uncertainties and their resolution during substitution. (Rott, Specht, & Knipping, 2021b) Similarly, show that actual problem-solving processes contain cycles and departures from a predetermined sequence. S2's return from looking back to calculation therefore supports using (Polya, 1973) stages as analytical reference points rather than as an irreversible progression. The shared iterative pattern may arise because evaluating a result supplies information that changes how earlier work is interpreted.

The implementation findings also reveal a difference between selecting an appropriate strategy and carrying it out accurately. (Güner & Erbay, 2021) found that eighth-grade students often had weak habits of checking, detecting, and correcting errors despite their own perceptions. S2's initial error and earlier confidence are compatible with this discrepancy, whereas the later verification of every equation provides evidence of a more thorough check. Using clinical interviews with prospective mathematics teachers, (Kaya & Kepceoglu, 2022) identified several forms of metacognitive failure. The present cases add an account of how an inconsistency can be followed by recovery, but do not demonstrate the absence of such failures. Differences in educational level, problem structure, participant selection, and interview prompts may explain why complete verification became visible here.

These comparisons reinforce the need to distinguish statements about checking from evidence that checking occurred. (Jacobse & Harskamp, 2012) Compared approaches to measuring metacognition in fifth-grade mathematical problem solving and highlighted limitations of convenient self-report measures. Their measurement focus differs from the

present reconstruction of written work and interview accounts, but both approaches require evidence beyond general confidence. (Flavell, 1979) Distinguishes metacognitive knowledge and experiences from the goals and actions associated with cognition; (Veenman, Van Hout-Wolters, & Afflerbach, 2006) discuss related conceptual and methodological distinctions. Accordingly, S2's initial confidence was insufficient to establish validity. The stronger evidence was the explanation of the failed equation, identification of the incorrect subtraction, and verification after correction.

Confidence in these cases should also be distinguished from general beliefs and attitudes toward mathematics. (Yorulmaz, Uysal, & Çokçalışkan, 2021) found an association between metacognitive awareness and problem-solving beliefs in a survey of Turkish pre-service primary teachers. (Demirel, Derman, & Karagedik, 2015) investigated relationships between reflective thinking toward problem solving and attitudes toward mathematics. The present findings are compatible with attention to learners' judgments, but concern confidence in one particular solution after checking. They cannot establish a relationship between clarification and a general attitudinal trait. This difference follows partly from measurement: inventories describe broader tendencies, while task-based interviews reveal the mathematical grounds invoked for a particular judgment.

Looking back produced two distinct consequences. S1's verification confirmed an existing answer; S2's verification exposed an inconsistency and required correction. These outcomes elaborate the monitoring and evaluation functions discussed (Geng, 2011; Schneider & Artelt, 2010). (Nur et al., 2026) review identifies evaluation as a challenging dimension within the literature on metacognition and mathematical problem solving. The present cases illustrate how evaluation may become mathematically explicit, but their purposive selection precludes conclusions about how common successful checking is. The consistency with earlier work concerns the role of scrutiny and regulation, rather than evidence that all learners complete these processes effectively.

The mathematical structure of the task helps explain why checking all conditions mattered. S2's initial values satisfied equations A, B, and C while failing equation D. Because subsequent variables had been calculated from the incorrect value of x_3, local consistency within the substitution chain did not establish a solution of the entire system. Rechecking the original constraints exposed the error and required recalculating every affected value. This provides a concrete explanation for the interaction between monitoring and regulation described by (Jacobse & Harskamp, 2012; Melanie G. Gurat, 2016). It also suggests that differences between studies may reflect how readily a task exposes an inconsistency and whether its quantities depend on earlier results.

The corrective pathway has an educational connection with research on learning from unsuccessful attempts, although the designs and claims differ. (Kapur, 2014) showed benefits of productive failure followed by instruction for conceptual understanding and transfer. (Baumgartner et al., 2025) reported promising results for problem solving before instruction in university linear algebra, while acknowledging the limits of a design without a control group. In the present study, S2 already used an appropriate method and corrected its arithmetic implementation; there was no productive-failure intervention or assessment of learning gains. The comparison therefore supports examining unsuccessful work as a possible resource for learning, without equating a single correction episode with demonstrated conceptual development.

Recent reviews provide further context for the findings. (Mannan et al., 2026) examine GeoGebra-supported metacognition, while (Mafarja et al., 2026) synthesize approaches to developing prospective mathematics teachers' mathematical thinking. Their broader educational concerns are compatible with making reasoning and evaluation visible. The present study identifies clarification in written work and interviews without a digital intervention. This difference suggests that the relevant analytical questions can be posed across learning environments: what caused uncertainty, what mathematical grounds were consulted, and what changed after checking? However, the cases provide no evidence for comparing technological

and nontechnological support or determining which environment develops clarification more effectively.

The study's contribution is a case-grounded distinction between confirmatory and corrective clarification within the same task and solution method. (Hidajat et al., 2019) foreground sources of perplexity, and (Aras et al., 2022) examine scaffolding within reflective problem solving. The present analysis extends that focus by following the consequences of clarification: preserving a justified answer or revising incorrect values and checking their dependencies. In relation to (Jacobse & Harskamp, 2012) work, it also specifies the mathematical evidence underlying a judgment of correctness. These categories offer an analytical perspective for examining doubt, checking grounds, and outcomes; they remain provisional categories derived from two cases, rather than a validated typology or a claim that confirmation and correction have not previously been studied.

For classroom learning, the findings suggest using tasks that require students to explain uncertainty, identify the first inconsistent step, and justify each affected correction. (Tachie, 2019a) describes reflective classroom discourse among student teachers, while (Tachie, 2019b) examines the application of metacognitive skills and strategies in mathematical problem solving. These studies support opportunities to discuss reasoning and evaluate solutions. (Özsoy, G., & Ataman, 2009) provide intervention evidence that explicit metacognitive strategy training can improve school students' problem-solving achievement. Drawing on these studies, teacher educators could ask learners to compare a valid solution with one containing a propagating error and explain why checking only selected equations is insufficient. Such activities are recommendations arising from the cases and literature; their effectiveness requires direct evaluation.

Assessment should recognize the reasons behind verification as well as the final answer. A rubric could examine whether a prospective teacher identifies the uncertain aspect, selects a relevant mathematical check, explains its outcome, and revises all affected results when necessary. (Choy, 2012) Found limited depth in teachers' reflection on teaching practices, indicating that reflection should not be assumed from participation alone. (Lit, Kool, & Drijvers, 2026) emphasize teachers' interpretation of students' work and facilitation of mathematical discussions. (Schoenfeld, 2020) similarly places mathematical practices within classroom activity. The present findings offer examples for assessing mathematical clarification, although transfer from personal solution checking to interpreting pupils' reasoning has yet to be demonstrated.

At the curriculum level, teacher education programs could incorporate reasoned verification within linear algebra coursework and problem-solving pedagogy. (Artigue & Houdement, 2007) discuss the relationship between didactic perspectives and curricular choices in problem solving. (Giyarti et al., 2025; Jiang et al., 2022) show the relevance of investigating prospective teachers' understanding of problem solving and its teaching. Together with the present cases, this literature provides a rationale for course activities that make uncertainty and justification discussable. The findings can inform local curriculum development and assessment criteria, but do not warrant a system-wide policy mandate or demonstrate improved teaching performance.

The methodological scope limits the contribution. Two purposively selected participants completed one traffic-flow task using substitution, and S2's corrective account was elicited in a follow-up interview. Differences in the timing and amount of questioning mean that the cases are not equivalent observations of entirely spontaneous checking. Consistent with case-study guidance from (Baxter & Jack, 2015; Crowe et al., 2011), transferability depends on the similarity of task, participants, and setting. (Nowell, Norris, White, & Moules, 2017) emphasize transparent links between qualitative evidence and interpretations. Accordingly, the two categories should remain traceable to the written work and interview excerpts; this study cannot estimate their prevalence, establish causal instructional effects, or explain the contrast through unmeasured personal characteristics.

Further research should examine whether confirmatory and corrective clarification recur across mathematical topics, solution methods, and participant groups. (Schoenfeld, 2020)

combine conceptualizations of metacognitive knowledge to analyze students' accounts of mathematical problem solving, suggesting possibilities for connecting clarification with task, strategy, and conditional knowledge. (Veenman et al., 2006) provide a methodological basis for distinguishing knowledge about regulation from regulation enacted during a task. Studies could therefore combine contemporaneous verbal protocols with written records, distinguish independent checks from prompted ones, and follow participants over several tasks. Instructional studies should test whether activities focused on uncertainty, error dependencies, and full-condition verification strengthen conceptual understanding, reasoned judgment, or later teaching practice.

CONCLUSION

The two cases show that reflective clarificative thinking in solving systems of linear equations can be traced through perplexity, identification of the uncertain aspect, mathematically justified checking, and the consequences of clarification. Across Polya's stages, checking concerned flow directions and diagram information, model adequacy and substitution order, operations and intermediate results, and the consistency of the solution with all original equations. A correct answer or a statement of confidence therefore requires evidence of checking actions and their reasons to support an interpretation of reflective thinking. The comparison revealed confirmatory clarification in S1 and corrective clarification in S2. S1 confirmed an initially correct solution by substituting the values into every equation. S2 found that $x_3 = 330$ did not satisfy equation D, traced the subtraction error in $820 - 470$, corrected x_3 to 350, and recalculated $x_2 = 230$, $x_1 = 280$, and $x_4 = 590$. The follow-up interview showed that S2 verified all four equations before expressing confidence in the solution. Substitution was retained; the calculated values changed. In both cases, mathematical certainty emerged through reasoned checking that either maintained or revised the answer. These findings suggest that instruction on systems of linear equations should encourage students to articulate doubts, detect errors, explain their effects on subsequent results, and verify all problem conditions after correction. Looking back should allow students to revisit earlier calculations. Further research could examine confirmatory and corrective clarification across additional tasks and participants and compare independently performed corrections with those elicited through interview questions.

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